An Improved Autotune Identification Method

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A method is proposed to estimate accurately the model parameters of a first order plus time delay (FOPTD) transfer function model using the conventional relay autotune method. Usually, the value of the delay is assumed or noted from the initial portion of the response of the system. Whenever identifying a higher order dynamics system by a FOPTD model, this method identifies wrongly the time constant as negative (Li et al., 1991) due to the error in identifying the time delay which is due to error in the model structures. In the present work, an additional equation is formulated to calculate accurately the parameters of the FOPTD model. Even when the actual system is FOPTD and the time delay to time constant ratio is larger, then higher order harmonics can not be neglected in the output response. Also in the present work, a method is proposed to consider the higher order harmonics of the relay oscillations. Improved accurate values for the controller ultimate gain are obtained. The estimated model parameters of a FOPTD model are compared with that obtained by Li et al. (1991) and that with the exact model parameters of FOPTD systems. The performance of the controller designed on identified model is compared with that identified by Li et al. (1991). The proposed method gives a better performance.

Keywords:
Relay tuning, higher order harmonics, FOPTD, single test

Introduction

Astrom and Hagglund (1984) have suggested the use of an ideal (on-off) relay to generate a sustained oscillation in the closed loop. The amplitude (A) and the period of oscillation (\(p_u\)) are noted from the sustained oscillation. The controller ultimate gain is calculated from \(k_u = 4h/(\pi A)\), where h is the relay height and ‘A’ is the process output amplitude. If the relay output is expanded in a Fourier series then it is possible that there is a considerable power present in the higher order harmonics. In deriving the equation for the ultimate gain of the controller, it is assumed that higher order harmonics of the relay output are filtered by the system and allows only the sine wave with the fundamental frequency of oscillation (Astrom and Hagglund, 1984). If the system has a low pass filter characteristics at the ultimate frequency, then the assumption is valid and the method gives an accurate estimate of the ultimate gain.

Luyben (1987) has used the relay feedback method to identify the model parameters (\(k_u\), \(\tau\) and \(D\)) of a first order plus time delay model. Using the controller ultimate gain and the period of oscillation, two equations are formulated using the amplitude criterion and phase angle criterion. The value of the time delay is noted from the initial portion of the relay output. The values of \(k_u\) and \(\tau\) are then calculated from the derived two equations. For identifying FOPTD model for a system whose dynamics are of really higher order, then the method of Luyben gives negative time constant (Li et al., 1991). The error lies in the difference in the model structures.

Further when the ratio of time delay to time constant is larger, Li et al. (1991) have reported that the model identified by relay autotune method gives as high as 27 % error in the value of \(k_u\). For smaller \(D/\tau\), Li et al. (1991) have found by simulation that the relay auto tune method gives \(-18\) % error in the value of \(k_u\). The auto tune method uses only principle harmonics for the calculation of model parameters. Li et al. (1991) and Leva (1994) have proposed the use of two relay feedback test (one conventional relay test and the second with added known delay). In these methods all the parameters of the model are identified. Shen et al. (1996) have used a biased relay for getting the model parameters using a single relay test. Huang et al. (1996) have proposed an asymmetrical relay method. Yu (1999) has given an excellent survey of relay feedback method. Park et al.(1997) have proposed modifications of the method under load disturbances. Consideration of higher order dynamics is given by Chiang et. al. (1992), Scali et. al. (1999), Luyben (2001), Kaya and Atherton (1999)
and Atherton (2000), Scali et. al. (1999) have proposed the use of several relay tuning tests to be carried out. Luyben method (2001) is not effective for unstable systems. In the methods proposed by Atherton (2000) and Kaya and Atherton (1999), computational complexity are involved. In the present work, (i) a modification in analyzing the data of the conventional relay method is proposed by formulating an additional equation so as to estimate more accurate value of the process gain \( k_p \) and hence more accurate FOPTD model parameters and (ii) a method to consider higher order harmonics in analyzing the conventional relay testing, for getting improved model parameters, is also proposed.

**Estimation of Time delay**

Recently Padmasree and Chidambaram (2001) have proposed a method for formulating an equation for time delay of the FOPTD model when analyzing the closed loop response of a PID controlled system. In the present work, their method in formulating the additional equation is applied here for analyzing the conventional relay feedback oscillation.

Let \( G(s) \) represent the process transfer function of the system

\[
G(s) = \frac{y(s)}{u(s)}
\]  

From the definition of Laplace transform we get:

\[
y(s) = \int y(t) \exp(-st) \, dt
\]  

Let us evaluate the above integral (from \( t = 0 \) to \( \infty \)) for \( s_1 = 8/t_o \), where \( t_o \) is the time at which few (say 3) repeated cycles of oscillation appear in the output. The reason for taking \( s_1 = 8/t_o \) is that for \( t > t_o \), because of very small value of the term [\( \exp(-st) \)], all further contribution by the subsequent terms is negligible while evaluating the integral value. Now we have the numerical value for \( y(s_1) \). Similarly from the \( u(t) \) versus time data, \( u(s_1) \) is calculated. From the assumed FOPTD model also, we can formulate the equation for \( y(s_1) \) as:

\[
y(s_1) = u(s_1) k_p \exp(-Ds_1)/\tau s_1 + 1
\]  

Thus we are able to formulate the above additional equation using \( y(s_1) \) and \( u(s_1) \). It is to be noted that we have to record the transient data of the relay till 2 or 3 sustained oscillations are obtained in the process output. From the amplitude criterion and phase angle criterion, we get the following equations for the parameters of FOPTD model as:

\[
\tau = \frac{v}{\omega_u}
\]

\[
D = \frac{[\pi - \tan^{-1}(v)]}{\omega_u}
\]

Using the above two equations in eq(3) we get:

\[
k_p \exp\left[\frac{(-s_1/\omega_u)[\pi - \tan^{-1}(v)]}{[v s_1/\omega_u + 1]}\right] = y(s_1)/u(s_1)
\]  

(6)

where \( v = [(k_u k_p)^2 - 1]^{0.5} \) and \( k_u \) is the controller ultimate gain.

Using the values of \( y(s_1) \) and \( u(s_1) \) in the above equations and solving the resulting nonlinear algebraic equation we get the value of \( k_p \). Using the value of \( k_u \) in Eqs(4) and (5), we get the values of the FOPTD model parameters \( \tau \) and \( D \). Let us consider an example given by Li et al. (1991)

\[
y(s)/u(s) = \exp(-2s)/[(10s + 1)(s + 1)]
\]  

(7)

Using the relay feedback method (with relay height as 1), we get the oscillations as shown in Fig 1. The steady oscillations show a pure sinusoidal wave form, thereby assumption of relay method to neglect higher order harmonics is valid for this example. By the Luyben (1987) method, the delay is noted as 2 from the initial response and from the oscillations it is noted that \( \omega_u = 0.5978 \) and \( \alpha = 0.1943 \). Using the relation \( k_u = 4h/(\pi A) \), we get \( k_u = 6.553 \).

![Fig. 1 — Relay feedback oscillations for the system \( \exp(-2s)/[(10s + 1)(s + 1)] \) relay height = 1](image)

Using the amplitude criterion and the phase angle criterion for a FOPTD system, Li et. al. (1991) have reported the values of \( k_p \) as -0.501 and \( \tau \) as -5.03. Since negative values for \( \tau \) and \( k_p \) are obtained, it is suggested by Li et al. (1991) that FOPTD model is not valid for this system and Li et al. (1991) fitted a second order plus time delay (SOPTD) model using a second relay test with a known additional delay time.

By the proposed method of analyzing the conventional relay method, we note from Fig 1 the
value of $t_s$ as 68.9 and hence using Eq (2) we get $y(s_j) = -0.1239$ for the calculated value of $u(s_j) = -0.0407$. Using Eqs (6), (4) and (5), we get $k_5 = 1.08$, $\tau = 11.725$ and $D = 2.87$. The proposed method is able to identify a FOPTD model. The open loop response of the actual system and the identified FOPTD model is shown in Fig 2. A good matching is obtained. Fig 2 shows that the response of the identified FOPTD model by the present method is better than that of SOPTD model proposed by Li et al. (1991). Thus by properly interpreting the conventional relay oscillations, we can estimate the model parameters of a FOPTD model.

**Proposed method for considering higher order harmonics**

From the Fourier series analysis, it can be easily shown (Astrom and Hagglund, 1984) that a relay consists of many sinusoidal waves of odd multiples of fundamental frequency ‘$\omega$’ and with the amplitude $4hn/(n\pi)$ ($n = 1, 3, 5, \ldots$). The input to the process thus consists of many sine waves. For a FOPTD system, output wave is also a sinusoidal wave with different amplitude and different frequency. The oscillations what we observe i.e., $y(t)$ is the additional of many of sine waves:

$$y(t) = [A_1 \sin(\omega_u t + \phi_1) + (1/3) A_3 \sin(3\omega_u t + \phi_3) + (1/5) A_5 \sin(5\omega_u t + \phi_5) + \ldots]\quad(8)$$

where

$$\begin{align*}
A_1 &= 1/[1 + (\tau\omega_u)^2]; \\
A_3 &= 1/[1 + (3\tau\omega_u)^2]; \\
A_5 &= 1/[1 + (5\tau\omega_u)^2], \ldots \text{etc.} \quad(9)
\end{align*}$$

$\phi_1 = -D\omega_u - \tan^{-1}(\tau\omega_u) = -\pi \quad(10a)$

$\phi_3 = -3D\omega_u - \tan^{-1}(3\tau\omega_u)$;

$\phi_5 = -j D\omega_u - \tan^{-1}(5\tau\omega_u)$

$y(t) = A_1 [\sin(\omega_u t + \phi_1) + (1/3) b_3 \sin(3\omega_u t + \phi_3) + (1/5) b_5 \sin(5\omega_u t + \phi_5) + \ldots]\quad(11)$

where

$$\begin{align*}
b_3 &= \{(1 + (\tau\omega_u)^2)/[1 + (3\tau\omega_u)^2]\}^{0.5}; \\
b_5 &= \{(1 + (\tau\omega_u)^2)/[1 + (5\tau\omega_u)^2]\}^{0.5}; \text{ etc.,} \quad(12)
\end{align*}$$

If $\tau\omega_u$ is assumed very large ($\infty$), then $y(t)$ will consist of only the fundamental harmonics. In Eq (8), the terms containing $A_3 \sin(3\omega_u t + \phi_3), A_5 \sin(5\omega_u t + \phi_5)$ etc. will be neglected. As stated earlier, this assumption gives a large error in the calculation of $k_5$ and hence in the estimated model parameters of FOPTD model. In what follows we will consider higher order dynamics for the calculation of the model parameters.

Let us derive approximate evaluation of $y(t)$ for the limiting cases of smaller $\tau\omega_u$ and separately for larger $\tau\omega_u$. Whenever the relay oscillation is close to rectangular wave form, the results for smaller $\tau\omega_u$ is to be used. If the relay oscillation is closer to triangular wave form, then the following result for larger $\tau\omega_u$ is to be used. If the relay oscillation is close to sine wave form, then the standard equation considering only the fundamental frequency of oscillation can be used.

**case 1:** when $\tau\omega_u$ is smaller

$$\phi_3 = -3D\omega_u - \tan^{-1}(3\tau\omega_u) \quad(13a)$$

$$= -3D\omega_u - 3 \tan^{-1}(\tau\omega_u) = 3\phi_1 = -3\pi \quad(13b)$$

Similarly,

$$\phi_5 = -5\pi; \phi_N = -N\pi \quad(13c)$$

Hence Eq (11) can be written as:

$$y(t) = A_1 [-\sin(\omega_u t) - (1/3) b_3 \sin(3\omega_u t) - (1/5) b_5 \sin(5\omega_u t) + \ldots]\quad(14)$$

For smaller value of $2\omega_u$, $\tau\omega_u$ can be neglected when compared to 1 and hence the values of $b_1, b_3, \ldots b_N$ can each be approximated to 1. Hence Eq. (14) becomes:

$$y(t) = A_1 [-\sin(\omega_u t) - (1/3) \sin(3\omega_u t) - (1/5) \sin(5\omega_u t) + \ldots]\quad(15)$$

Here the value of ‘$A_1$’ is to be calculated. The value of ‘$A_1$’ is not the amplitude what is observed from the output oscillation.
Li et al. (1991) have assumed that the system is a low pass filter so that higher order harmonics of the input sine waves are filtered and the system oscillation contains only the fundamental frequency. The value of \( k_u \) is obtained by Li et. al. (1991) from

\[
k_u = 4h/(A_o\pi)
\]

(16)

where \( A_0 \) is the observed output amplitude. In Eq (15), when only the first term is considered then \( A = A_o \). Since \( y(t) \) contains many sine waves and can not be represented by single sine wave of amplitude of \( A_o \), the evaluated \( k_u \) deviates from the actual value. Li et. al. (1991) have reported that an error of –18 % to 27 % is noted in the estimated value of \( k_u \) by using Eq (16).

This problem can be overcome as follows: From the output oscillations, it is possible to calculate \( y(t) \) at any time ‘t’. ‘\( \omega_u \)’ is the frequency of observed oscillations. From Eq (15), we get

\[
A_1 = y(t) / \Sigma [\sin(i\omega_u t)/i]
\]

(17)

Let us consider the time (\( t^* \)) at which

\[
\omega_u t^* = 0.5\pi.
\]

(18)

Then Eq (15) becomes:

\[
y(t^*) = A [1 – (1/3) + (1/5) – \ldots] \]

(19)

where \( A = \) modulus of \( A_1 \). Let the number of terms to be considered in the above equation is denoted as \( N \). Using the limiting value for the summation term (0.25\( \pi \)), we get from Eq (19):

\[
A = 1.273 \ y(t^*)
\]

(20)

The time at which the value of \( y(t^*) \) is to be noted from the response is given by

\[
t^* = 0.5 \pi/\omega_u
\]

(21)

From the relay oscillation test, we can note down the value of \( \omega_u \) and \( t^* \) can be calculated and the value of \( y(t^*) \) is noted from the process output. The value of \( k_u \) is given by

\[
k_u = 4h/(A\pi)
\]

(22)

The method is tested on FOPTD systems with various values of \( D/\pi \) (refer to Table 1, for model parameters) and the results on \( k_u \) by the present method given in Table 2a along with the results of Li et al. (1991). From the identified values of \( k_u \) and the ultimate frequency of oscillation, the parameters of a FOPTD are evaluated. Table 1 gives the values of identified model parameters compared with the actual model parameters for different case studies of larger delay to time constant ratio.

If the system allows all the frequencies of oscillations (in that case, the output oscillation is also a rectangular wave form), then the limiting case of \( \pi/4 \) can be used in the above analysis in order to estimate the model parameters. However, if the process filters out beyond certain frequencies, then the output will very much deviate from the rectangular wave form. In that case, we have to use an appropriate value of \( N \) in order to get accurate values of the FOPTD model parameters.

Eq (19) shows that the sum of terms in the bracket varies from 1 to 0.8 for \( N = 1 \) to \( \infty \). Hence, an error of 1.27 times of actual value is obtained when \( N = 1 \) is used instead of \( N = \infty \). The number of terms to be considered in Eq (19) is any one of 1,3,5,7,9,… and \( \infty \). When the number of terms in Eq (19) is \( \infty \), then Eq (20) is considered for ‘A’. Li et. al. (1991) have used \( N = 1 \) (i.e., \( A = A_o \)). The present study shows that the value of \( N = 5 \) gives a better result on the calculated \( k_u \) and on the model.

| Table 1 – Comparison of estimated \( k_u \) and identified FOPTD model parameters for the present method (when \( \omega t = \pi/2 \) and Li et al. (1991) method) |
|---|---|---|---|---|---|---|---|---|---|
| Simulated Model Parameters | Ultimate gains | Freq. From osc | Identified Model Parameters |
| No | \( k_p \) | \( \tau \) | \( D \) | \( k_u \) (ana) | \( k_u \) (Li) | \( kD_u \) (Pro) | \( \omega \) (osc) | \( k_p \) (Li) | \( k_p \) (Pro) | \( \tau \) (Li) | \( \tau \) (Pro) |
| 1 | 1 | 0.2 | 4 | 1.01 | 1.27 | 1.00 | 0.76 | 0.79 | 1.00 | 0.14 | 0.14 |
| 2 | 1 | 0.4 | 4 | 1.04 | 1.27 | 1.068 | 0.74 | 0.80 | 0.98 | 0.28 | 0.42 |
| 3 | 1 | 0.5 | 4 | 1.06 | 1.27 | 1.077 | 0.72 | 0.82 | 0.97 | 0.37 | 0.45 |
| 4 | 1 | 1.0 | 4 | 1.19 | 1.30 | 1.18 | 0.67 | 0.86 | 0.96 | 0.74 | 0.81 |
| 5 | 1 | 1.0 | 5 | 1.13 | 1.28 | 1.13 | 0.55 | 0.83 | 0.96 | 0.73 | 0.78 |

The time delay values for the present method are 4.133, 3.83, 3.93, 3.94, 4.95 respectively for No 1 to 5.
parameters of FOPTD. Fig 3 shows the system oscillations (both the dynamics and steady-state) of relay test for the system \( \exp(-4s)/(s/c_116)^1 \) with \( \tau = 1 \) and 0.4 (i.e., \( D/\tau = 4 \ & 10 \)). If the system’s steady-state oscillation resembles that of pure sinusoidal wave (as shown in Fig 1), then a value of \( N = 1 \) can be used. If the oscillation deviates from pure sine wave and depending on the extent of deviation, \( N = 3 \) or 5 or 7 can be considered. For example for the Fig 3a, a value of \( N = 3 \) or 5 can be considered. When the oscillation is near to that of rectangular then a value of \( N = 9 \) can be considered (example for the condition of Fig 3b). An empirical relation between \( D/\tau \) versus \( N \) is fitted as

\[
N = -0.0395(D/\tau)^2 + 1.4746(D/\tau) - 1.9266 \quad (23)
\]

The nearest integer value of \( N \) from the above equation is to be used.

In the later section we will show, by a simulation study on a higher order system, the improved

| Table 2a – Effect of including the higher order harmonics on \( k_u \) (Eqs. 19 & 22) |
|-----------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( D/\tau \)     | \( k_u \) | \( k_u \) | \( k_u \) | \( k_u \) | \( k_u \) | \( k_u \) | \( k_u \) |
| 4               | 1.30      | 1.220     | 1.175     | 1.155     | 1.145     | 1.106     | 1.19      |
| 5               | 1.28      | 1.172     | 1.128     | 1.109     | 1.098     | 1.061     | 1.13      |
| 8               | 1.27      | 1.118     | 1.077     | 1.059     | 1.050     | 1.013     | 1.06      |
| 10              | 1.27      | 1.109     | 1.068     | 1.051     | 1.040     | 1.005     | 1.04      |
| 20              | 1.27      | 1.10      | 1.06      | 1.04      | 1.03      | 1.02      | 1.01      |

| Table 2b – Details of calculation for Table 2a |
|-----------------|-----------|-----------|-----------|-----------|
| \( D/\tau \)     | \( A_0 \) | \( \omega_u \) | \( t^* \) | \( Y(\theta) \) |
| 4               | 0.9817    | 0.67      | 2.34      | 0.904     | 1.089 |
| 5               | 0.9933    | 0.69      | 2.28      | 0.942     | 1.127 |
| 8               | 0.9997    | 0.72      | 2.17      | 0.987     | 1.182 |
| 10              | 1.0000    | 0.76      | 2.14      | 0.995     | 1.225 |
| 20              | 1.0000    | 0.76      | 2.08      | 1.00      | 1.273 |

| Table 2c – Effect of including the higher order harmonics on \( k_u \) (use of Eqs. 33&22) |
|-----------------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( D/\tau \)     | \( k_u \) | \( k_u \) | \( k_u \) | \( k_u \) | \( k_u \) | \( k_u \) |
| 0.2              | 7.023     | 8.73      | 8.98      | 9.09      | 9.15      | 9.36      | 8.50      |
| 0.08             | 16.64     | 19.78     | 20.34     | 20.58     | 20.72     | 21.21     | 20.28     |
| 0.04             | 32.39     | 38.06     | 39.15     | 39.61     | 39.88     | 40.82     | 39.98     |

| Table 2d – Details of calculation for Table 2c |
|-----------------|-----------|-----------|-----------|-----------|
| \( D/\tau \)     | \( A_0 \) | \( \omega_u \) | \( t^* \) | \( Y(\theta) \) |
| 0.2              | 0.1813    | 0.4274    | 3.67      | 0.1678    | 0.1458 |
| 0.08             | 0.0765    | 0.408     | 3.85      | 0.0741    | 0.0644 |
| 0.04             | 0.0391    | 0.4002    | 3.925     | 0.0385    | 0.0334 |

* using first three terms in the series

\( * \) using first three terms in the series

F i g. 3 – Relay feedback oscillations for the system (a) \( \exp(-4s)/(s+1) \), (b) \( \exp(-4s)/(0.4s+1) \) relay height = 1
estimate of $k_u$ and hence on the identified model parameters of a FOPTD system. Now let us consider the limiting case of larger $\tau_0$, for calculation of $k_u$.

**case 2:** limiting case of larger $\tau_0$

$$\phi_1 = -D\omega_u - \tan^{-1}(\tau_0\omega_u) = -\pi \tag{24a}$$

$$= -D\omega_u - (\pi/2) = -\pi \tag{24b}$$

hence

$$D\omega_u = 0.5\pi \tag{25}$$

$$\phi_3 = -3D\omega_u - \tan^{-1}(3\tau_0\omega_u) \tag{26a}$$

$$= -3(D\omega_u) - \pi/2 = -4\pi/2 \tag{26b}$$

Similarly,

$$\phi_2 = -6\pi/2, \ldots; \phi_N = -(N + 1)\pi/2 \tag{27}$$

Hence Eq (11) can be written as:

$$y(t) = A_1[-\sin(\omega_0 t) - (1/3) b_3 \sin(3\omega_0 t) - \ldots] \tag{28}$$

where

$$b_3 = \{1 + (\tau_0\omega_u)^2\}/[1 + (3\tau_0\omega_u)^2]\}^{0.5};$$

$$b_5 = \{1 + (\tau_0\omega_u)^2\}/[1 + (5\tau_0\omega_u)^2]\}^{0.5}$$

For larger value of $\tau_0$, 1 can be neglected when compared to the value of $\tau_0\omega_u$ and hence the values of $b_1, b_3, \ldots$ can be approximated as:

$$b_3 = 1/3; b_5 = 1/5; b_7 = 1/3, \ldots \tag{29}$$

Hence Eq (28) becomes:

$$y(t) = A_1[-\sin(\omega_0 t) + (1/9) \sin(3\omega_0 t) - \ldots] \tag{30}$$

Here the value of ‘$A_1$’ is to be calculated. The value of ‘$A_1$’ is not the amplitude what is observed from the output oscillation.

Li et al. (1991) have reported that an error of -19% is noted in the estimated value of $k_u$ by using Eq (22) for larger values of $\tau_0\omega_u$. This problem can be overcome as follows: From the output oscillations, it is possible to calculate $y(t)$ at any time ‘$t$’. ‘$\omega_0$’ is the frequency of observed oscillations. From Eq (30), we get

$$a = y(t)/\Sigma [\sin(\omega_0 t)/t^2] \tag{31}$$

Let us consider the time ($t^*$) at which

$$\omega_0 t^* = 0.5\pi. \tag{32}$$

Then Eq (30) becomes:

$$y(t^*) = A_1[1 + (1/9) + (1/25) + (1/49) + (1/81) + \ldots] \tag{33}$$

Let the number of terms to be considered in the above equation denoted as $N$. Using the limiting value for the summation term $(0.125\pi^2)$, we get from Eq (33):

$$A = 0.810 \ y(t^*) \tag{34}$$

The time at which the value of $y(t^*)$ is to be noted from the response is given by

$$t^* = 0.5 \ \pi/\omega_u \tag{35}$$

Table (2b) compares the values of $k_u$ from Eq (33) for different number of terms to be considered in Eq (33) for the lower values of $D/\tau$. The proposed method of considering the higher order harmonics is able to explain the reported maximum error of -18% and +27% as shown in Tables (2b) and (2a) respectively.

**Simulation study**

Let us consider an example with a higher order dynamics:

$$y(s)/u(s) = 1 \exp(-4s)/(0.5s + 1)^3 \tag{36}$$

Using the relay feedback method (with relay height as 1) we get an oscillation shown in Fig 4. From Fig 4, we noted from the output $\omega_u$ as 0.5889 and ‘$A$’ as 0.9969. Fig 4 shows that the oscillations contain a higher order harmonics (since the steady oscillation is not a pure sinusoidal wave form). From Fig 4, we note $t_u = 56$ and hence $s_1 = 8/56$. The value of $y(s_1)$ and $u(s_1)$ are calculated. 5 terms ($N = 5$) are used in evaluating the sum term of Eq (19). The calculated value of ‘$A$’ is 1.165. The identified value of $k_u$ is 1.093. Hence, the present method gives the model parameters of FOPTD as $k_p = 0.9714, D = 4.7061$ and $r = 0.605$. Li et al. (1991) have reported the identified value of $k_u$ as 1.28 and that of the
model parameters as $k_p = 1.0841$, $D = 4.04$ (noted from the initial response) and $\tau = 1.6362$. Li et al. (1991) assumed all higher order harmonics are filtered out by the system (Fig 4 shows that it is not so). The open loop response of the actual system and that of the identified FOPTD models are compared in the Fig 5. The response by the present model is better than that of Li et al. (1991). A PID controller is designed based on $k_c$ and $P_u$ by using Ziegler-Nichols continuous cycling tuning method. The PID settings by the present identified model are: $k_c = 0.656; \tau_1 = 5.34$ and $\tau_0 = 1.334$ and by Li et al model as $k_c = 0.768; \tau_1 = 5.34$ and $\tau_0 = 1.335$. The performances of the PID controllers are evaluated on the original transfer function model of the process. The performance of the PID controller based on the present model gives an improved performance as shown in Fig 6.

PID controllers are also designed based on the FOPTD model parameters by using Ziegler-Nichols open loop tuning formula. Since the ratio of delay to time constant is more than 1, the appropriate modified Ziegler-Nichols tuning formulae (Chidambaram, 1998) are used: $k_c = 0.6462 \left[1 + \left(\pi \tau/D\right)^2\right]^{0.5}$; $\tau_1 = D$ and $\tau_0 = 0.25 D$. The PID settings for the present identified model are $k_c = 0.73; \tau_1 = 4.70$ and $\tau_0 = 1.18$ whereas for the model identified by Li et al. (1991) are $k_c = 0.971; \tau_1 = 4.04$ and $\tau_0 = 1.01$. Fig 7 shows the performance comparison of the closed loop system evaluated on the original higher order system using the PID settings based on identified model. The performance of the present method gives a better response than that identified by Luyben (1987) method.

The effect of measurement noise is studied by adding a random noise (with mean $= 0$ and the standard deviation $= 0.5 \%$) to the process output and the corrupted signal is used for relay feedback. The present method gives $s_1 = 0.1344, \omega_n = 0.72, \nu_s = 0.2168, \nu_s = 0.3882$ and hence the identified model parameters of the FOPTD are obtained as $k_p = 1.005, D = 3.82$ and $\tau = 0.5$. These values are closer to the actual model parameters of $k_p = 1$, $D = 4.0$ and $\tau = 0.5$.

**Conclusions**

A method is suggested to formulate an additional equation so that the process gain can also be estimated using the conventional relay auto-tune method. This method avoids getting a negative time constant of a FOPTD model. For systems showing higher order harmonics in the response, a modifica-
tion of the calculation for the model parameters of FOPTD model using the conventional relay feedback method is also proposed. This method does not assume the complete filtering of higher order harmonics. The method of calculation is also simple. The present method gives an improved value for the controller ultimate gain. The method gives more accurate results [on $k_u$ and on the identified FOPTD model parameters] than that proposed by Luyben (1987) and Li et al. (1991). Simulation results show that the present method gives improved open loop and as well as closed loop performances.

**Nomenclature**

- $A_o$ – amplitude of oscillation observed from the process output
- $A_1$ – amplitude of oscillation corresponds to the principle harmonics calculated from Eq(22)
- $G$ – process transfer function
- $h$ – relay height
- $L$ – process delay
- $k_c$ – controller delay
- $k_p$ – process gain
- $k_u$ – controller ultimate gain
- $N$ – number of terms considered in Eq (19) or Eq (33)
- $r_u$ – period of output oscillation
- $s$ – Laplace variable
- $s_1 = 8/t_s$ – time taken to reach 3 invariant cycle of oscillation in the output
- $t$ – time
- $t^* = 0.5 \pi/\omega_u$ – time
- $u$ – input variable
- $v = [(k_c k_p)^2 - 1]^{0.5}$
- $y$ – output variable
- $\tau$ – process time constant
- $\tau_d$ – process time delay
- $\tau_i$ – integral time
- $\tau_D$ – derivative time
- $\omega$ – frequency of oscillation
- $\omega_u$ – ultimate frequency of oscillation

**References**